Chapter 2: Linear Functions

Chapter one was a window that gave us a peek into the entire course. Our goal was to understand the basic structure of functions and function notation, the toolkit functions, domain and range, how to recognize and understand composition and transformations of functions and how to understand and utilize inverse functions. With these basic components in hand we will further research the specific details and intricacies of each type of function in our toolkit and use them to model the world around us.

Mathematical Modeling

As we approach day to day life we often need to quantify the things around us, giving structure and numeric value to various situations. This ability to add structure enables us to make choices based on patterns we see that are weighted and systematic. With this structure in place we can model and even predict behavior to make decisions. Adding a numerical structure to a real world situation is called **Mathematical Modeling.**

When modeling real world scenarios, there are some common growth patterns that are regularly observed. We will devote this chapter and the rest of the book to the study of the functions used to model these growth patterns.

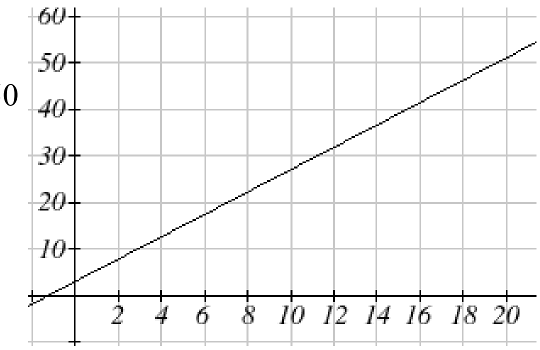
Section 2.1 Linear Functions

As you hop into a taxicab in Las Vegas, the meter will immediately read \$3.30; this is the "drop" charge made when the taximeter is activated. After that initial fee, the taximeter will add \$2.40 for each mile the taxi drives[1]. In this scenario, the total taxi fare depends upon the number of miles ridden in the taxi, and we can ask whether it is possible to model this type of scenario with a function. Using descriptive variables, we choose *m* for miles and *C* for Cost in dollars as a function of miles: *C(m)*.

We know for certain that $C(0) = 3.30$, since the \$3.30 drop charge is assessed regardless of how many miles are driven. Since \$2.40 is added for each mile driven, then
$C(1) = 3.30 + 2.40 = 5.70$
If we then drove a second mile, another \$2.40 would be added to the cost:
$C(2) = 3.30 + 2.40 + 2.40 = 3.30 + 2.40(2) = 8.10$

[1] http://taxi.state.nv.us/FaresFees.htm, retrieved July 28, 2010. There is also a waiting fee assessed when the taxi is waiting at red lights, but we'll ignore that in this discussion.

If we drove a third mile, another \$2.40 would be added to the cost:
$C(3) = 3.30 + 2.40 + 2.40 + 2.40 = 3.30 + 2.40(3) = 10.50$



From this we might observe the pattern, and conclude that if m miles are driven, $C(m) = 3.30 + 2.40m$ because we start with a \$3.30 drop fee and then for each mile increase we add \$2.40.

It is good to verify that the units make sense in this equation. The \$3.30 drop charge is measured in dollars; the \$2.40 charge is measured in dollars per mile. So

$$C(m) = 3.30 dollars + \left(2.40 \frac{dollars}{mile}\right)(m\ miles)$$

When dollars per mile are multiplied by a number of miles, the result is a number of dollars, matching the units on the 3.30, and matching the desired units for the C function.

Notice this equation $C(m) = 3.30 + 2.40m$ consisted of two quantities. The first is the fixed \$3.30 charge which does not change based on the value of the input. The second is the \$2.40 dollars per mile value, which is a **rate of change**. In the equation this rate of change is multiplied by the input value.

Looking at this same problem in table format we can also see the cost changes by \$2.40 for every 1 mile increase.

m	0	1	2	3
C(m)	3.30	5.70	8.10	10.50

It is important here to note that in this equation, the **rate of change is constant**; over any interval, the rate of change is the same.

Graphing this equation, $C(m) = 3.30 + 2.40m$ we see the shape is a line, which is how these functions get their name: **linear functions**

When the number of miles is zero the cost is \$3.30, giving the point (0, 3.30) on the graph. This is the vertical or *C(m)* intercept. The graph is increasing in a straight line from left to right because for each mile the cost goes up by \$2.40; this rate remains consistent.

In this example you have seen the taxicab cost modeled in words, an equation, a table and in graphical form. Whenever possible, ensure that you can link these four representations together to continually build your skills. It is important to note that you will not always be able to find all 4 representations for a problem and so being able to work with all 4 forms is very important.

Linear Function

A **linear function** is a function whose graph produces a line. Linear functions can always be written in the form

$f(x) = b + mx$ or $f(x) = mx + b$; they're equivalent

where

b is the initial or starting value of the function (when input, $x = 0$), and

m is the constant rate of change of the function

Many people like to write linear functions in the form $f(x) = b + mx$ because it corresponds to the way we tend to speak: "The output starts at b and increases at a rate of m."

For this reason alone we will use the $f(x) = b + mx$ form for many of the examples, but remember they are equivalent and can be written correctly both ways.

Slope and Increasing/Decreasing

m is the constant rate of change of the function (also called **slope**). The slope determines if the function is an increasing function or a decreasing function.

$f(x) = b + mx$ is an **increasing** function if $m > 0$

$f(x) = b + mx$ is a **decreasing** function if $m < 0$

If $m = 0$, the rate of change zero, and the function $f(x) = b + 0x = b$ is just a horizontal line passing through the point (0, b), neither increasing nor decreasing.

Example 1

Marcus currently owns 200 songs in his iTunes collection. Every month, he adds 15 new songs. Write a formula for the number of songs, *N*, in his iTunes collection as a function of the number of months, *m*. How many songs will he own in a year?

The initial value for this function is 200, since he currently owns 200 songs, so $N(0) = 200$. The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. With this information, we can write the formula: $N(m) = 200 + 15m$.

N(m) is an increasing linear function.

With this formula we can predict how many songs he will have in 1 year (12 months): $N(12) = 200 + 15(12) = 200 + 180 = 380$. Marcus will have 380 songs in 12 months.

Try it Now

1. If you earn \$30,000 per year and you spend \$29,000 per year write an equation for the amount of money you save after y years, if you start with nothing.
"The most important thing, spend less than you earn![2]"

Calculating Rate of Change

Given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2, or a set of points, $(x_1,\ y_1)$ and $(x_2,\ y_2)$, if we wish to find a linear function that contains both points we can calculate the rate of change, m:

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Rate of change of a linear function is also called the **slope** of the line.

Note in function notation, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example 2

The population of a city increased from 23,400 to 27,800 between 2002 and 2006. Find the rate of change of the population during this time span.

The rate of change will relate the change in population to the change in time. The population increased by $27800 - 23400 = 4400$ people over the 4 year time interval. To find the rate of change, the number of people per year the population changed by:

$$\frac{4400\,people}{4\,years} = 1100\frac{people}{year} = 1100 \text{ people per year}$$

Notice that we knew the population was increasing, so we would expect our value for m to be positive. This is a quick way to check to see if your value is reasonable.

Example 3

The pressure, P, in pounds per square inch (PSI) on a diver depends upon their depth below the water surface, d, in feet, following the equation $P(d) = 14.696 + 0.434d$. Interpret the components of this function.

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[2] http://www.thesimpledollar.com/2009/06/19/rule-1-spend-less-than-you-earn/

The rate of change, or slope, 0.434 would have units $\frac{\text{output}}{\text{input}} = \frac{\text{pressure}}{\text{depth}} = \frac{\text{PSI}}{\text{ft}}$. This tells us the pressure on the diver increases by 0.434 PSI for each foot their depth increases.

The initial value, 14.696, will have the same units as the output, so this tells us that at a depth of 0 feet, the pressure on the diver will be 14.696 PSI.

Example 4

If $f(x)$ is a linear function, $f(3) = -2$, and $f(8) = 1$, find the rate of change.

$f(3) = -2$ tells us that the input 3 corresponds with the output -2, and $f(8) = 1$ tells us that the input 8 corresponds with the output 1. To find the rate of change, we divide the change in output by the change in input:

$m = \frac{\text{change in output}}{\text{change in input}} = \frac{1-(-2)}{8-3} = \frac{3}{5}$. If desired we could also write this as $m = 0.6$

Note that it is not important which pair of values comes first in the subtractions so long as the first output value used corresponds with the first input value used.

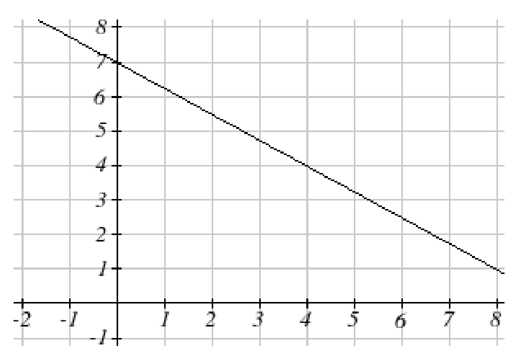
Try it Now

2. Given the two points (2, 3) and (0, 4), find the rate of change. Is this function increasing or decreasing?

We can now find the rate of change given two input-output pairs, and can write an equation for a linear function once we have the rate of change and initial value. If we have two input-output pairs and they do not include the initial value of the function, then we will have to solve for it.

Example 5

Write an equation for the linear function graphed to the right.



Looking at the graph, we might notice that it passes through the points (0, 7) and (4, 4). From the first value, we know the initial value of the function is $b = 7$, so in this case we will only need to calculate the rate of change:

$$m = \frac{4-7}{4-0} = \frac{-3}{4}$$

This allows us to write the equation:

$$f(x) = 7 - \frac{3}{4}x$$

Example 6

If $f(x)$ is a linear function, $f(3) = -2$, and $f(8) = 1$, find an equation for the function.

In example 3, we computed the rate of change to be $m = \frac{3}{5}$. In this case, we do not know the initial value $f(0)$, so we will have to solve for it. Using the rate of change, we know the equation will have the form $f(x) = b + \frac{3}{5}x$. Since we know the value of the function when $x = 3$, we can evaluate the function at 3.

$f(3) = b + \frac{3}{5}(3)$ Since we know that $f(3) = -2$, we can substitute on the left side

$-2 = b + \frac{3}{5}(3)$ This leaves us with an equation we can solve for the initial value

$b = -2 - \frac{9}{5} = \frac{-19}{5}$

Combining this with the value for the rate of change, we can now write a formula for this function:

$$f(x) = \frac{-19}{5} + \frac{3}{5}x$$

Example 7

Working as an insurance salesperson, Ilya earns a base salary and a commission on each new policy, so Ilya's weekly income, *I*, depends on the number of new policies, *n*, he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies, and earned \$920. Find an equation for *I(n)*, and interpret the meaning of the components of the equation.

The given information gives us two input-output pairs: (3,760) and (5,920). We start by finding the rate of change.

$$m = \frac{920-760}{5-3} = \frac{160}{2} = 80$$

Keeping track of units can help us interpret this quantity. Income increased by \$160 when the number of policies increased by 2, so the rate of change is \$80 per policy; Ilya earns a commission of \$80 for each policy sold during the week.

We can then solve for the initial value

$I(n) = b + 80n$	then when $n = 3$, $I(3) = 760$, giving
$760 = b + 80(3)$	this allows us to solve for b
$b = 760 - 80(3) = 520$	

This value is the starting value for the function. This is Ilya's income when $n = 0$, which means no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

Writing the final equation:

$I(n) = 520 + 80n$

Our final interpretation is: Ilya's base salary is \$520 per week and he earns an additional \$80 commission for each policy sold each week.

Flashback

Looking at Example 7:
Determine the independent and dependent variables.
What is a reasonable domain and range?
Is this function one-to-one?

Try it Now

3. The balance in your college payment account, C, is a function of the number of quarters, q, you attend. Interpret the function $C(a) = 20000 - 4000q$ in words. How many quarters of college can you pay for until this account is empty?

Example 8

Given the table below write a linear equation that represents the table values

w, number of weeks	0	2	4	6
$P(w)$, number of rats	1000	1080	1160	1240

We can see from the table that the initial value of rats is 1000 so in the linear format $P(w) = b + mw$, $b = 1000$.

Rather than solving for m, we can notice from the table that the population goes up by 80 for every 2 weeks that pass. This rate is consistent from week 0, to week 2, 4, and 6.

The rate of change is 80 rats per 2 weeks. This can be simplified to 40 rats per week and we can write
$P(w) = b + mw$ as $P(w) = 1000 + 40w$

If you didn't notice this from the table you could still solve for the slope using any two points from the table. For example, using (2, 1080) and (6, 1240),
$m = \dfrac{1240 - 1080}{6 - 2} = \dfrac{160}{4} = 40$ rats per week

Important Topics of this Section

Definition of Modeling
Definition of a linear function
Structure of a linear function
Increasing & Decreasing functions
Finding the vertical intercept (0, b)
Finding the slope/rate of change, m
Interpreting linear functions

Try it Now Answers

1. $S(y) = 30{,}000y - 29{,}000y = 1000y$ $1000 is saved each year.
2. $m = \dfrac{4-3}{0-2} = \dfrac{1}{-2} = -\dfrac{1}{2}$; Decreasing because $m < 0$
3. Your College account starts with $20,000 in it and you withdraw $4,000 each quarter (or your account contains $20,000 and decreases by $4000 each quarter.) You can pay for 5 quarters before the money in this account is gone.

Flashback Answers

n (number of policies sold) is the independent variable
I(n) (weekly income as a function of policies sold) is the dependent variable.

A reasonable domain is (0, 15)*
A reasonable range is ($540, $1740)*
*answers may vary given reasoning is stated; 15 is an arbitrary upper limit based on selling 3 policies per day in a 5 day work week and $1740 corresponds with the domain.

Yes this function is one-to-one

Section 2.1 Exercises

1. A town's population has been growing linearly. In 2003, the population was 45,000, and the population has been growing by 1700 people each year. Write an equation, $P(t)$, for the population t years after 2003.

2. A town's population has been growing linearly. In 2005, the population was 69,000, and the population has been growing by 2500 people each year. Write an equation, $P(t)$, for the population t years after 2005.

3. Sonya is currently 10 miles from home, and is walking further away at 2 miles per hour. Write an equation for her distance from home t hours from now.

4. A boat is 100 miles away from the marina, sailing directly towards it at 10 miles per hour. Write an equation for the distance of the boat from the marina after t hours.

5. Timmy goes to the fair with $40. Each ride costs $2. How much money will he have left after riding n rides?

6. At noon, a barista notices she has $20 in her tip jar. If she makes an average of $0.50 from each customer, how much will she have in her tip jar if she serves n more customers during her shift?

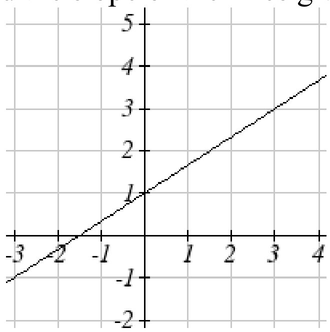
Determine if each function is increasing or decreasing

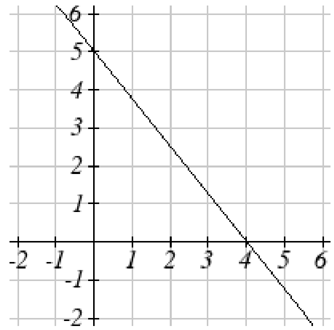
7. $f(x) = 4x + 3$
8. $g(x) = 5x + 6$
9. $a(x) = 5 - 2x$
10. $b(x) = 8 - 3x$
11. $h(x) = -2x + 4$
12. $k(x) = -4x + 1$
13. $j(x) = \frac{1}{2}x - 3$
14. $p(x) = \frac{1}{4}x - 5$
15. $n(x) = -\frac{1}{3}x - 2$
16. $m(x) = -\frac{3}{8}x + 3$

Find the slope of the line that passes through the two given points

17. (2, 4) and (4, 10)
18. (1, 5) and (4, 11)
19. (-1,4) and (5, 2)
20. (-2, 8) and (4, 6)
21. (6,11) and (-4,3)
22. (9,10) and (-6,-12)

Find the slope of the lines graphed

23. 

24. 

25. Sonya is walking home from a friend's house. After 2 minutes she is 1.4 miles from home. Twelve minutes after leaving, she is 0.9 miles from home. What is her rate?

26. A gym membership with two personal training sessions costs \$125, while gym membership with 5 personal training sessions costs \$260. What is the rate for personal training sessions?

27. A city's population in the year 1960 was 287,500. In 1989 the population was 275,900. Compute the slope of the population growth (or decline) and make a statement about the population rate of change in people per year.

28. A city's population in the year 1958 was 2,113,000. In 1991 the population was 2,099,800. Compute the slope of the population growth (or decline) and make a statement about the population rate of change in people per year.

29. A phone company charges for service according to the formula: $C(n) = 24 + 0.1n$, where n is the number of minutes talked, and $C(n)$ is the monthly charge, in dollars. Find and interpret the rate of change and initial value.

30. A phone company charges for service according to the formula: $C(n) = 26 + 0.04n$, where n is the number of minutes talked, and $C(n)$ is the monthly charge, in dollars. Find and interpret the rate of change and initial value.

31. Terry is skiing down a steep hill. Terry's elevation, $E(t)$, in feet after t seconds is given by $E(t) = 3000 - 70t$. Write a complete sentence describing Terry's starting elevation and how it is changing over time.

32. Maria is climbing a mountain. Maria's elevation, $E(t)$, in feet after t minutes is given by $E(t)=1200+40t$. Write a complete sentence describing Maria's starting elevation and how it is changing over time.

Given each set of information, find a linear equation satisfying the conditions, if possible

33. $f(-5)=-4$, and $f(5)=2$

34. $f(-1)=4$, and $f(5)=1$

35. Passes through (2, 4) and (4, 10)

36. Passes through (1, 5) and (4, 11)

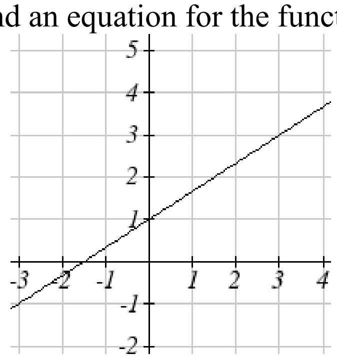
37. Passes through (-1,4) and (5, 2)

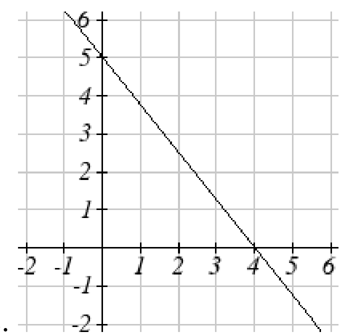
38. Passes through (-2, 8) and (4, 6)

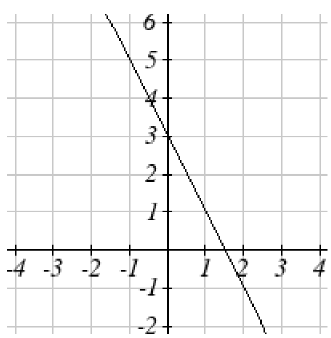
39. x intercept at (-2, 0) and y intercept at (0, -3)

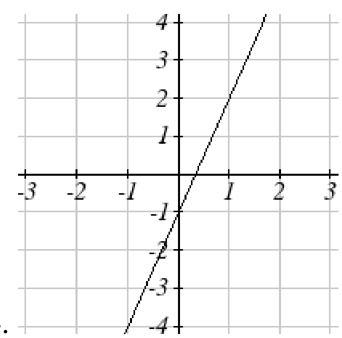
40. x intercept at (-5, 0) and y intercept at (0, 4)

Find an equation for the function graphed

41. 

42. 

43. 

44. 

45. A clothing business finds there is a linear relationship between the number of shirts, n, it can sell and the price, p, it can charge per shirt. In particular, historical data shows that 1000 shirts can be sold at a price of $\$30$, while 3000 shirts can be sold at a price of $\$22$. Find a linear equation in the form $p = mn + b$ that gives the price p they can charge for n shirts.

46. A farmer finds there is a linear relationship between the number of bean stalks, n, she plants and the yield, y, each plant produces. When she plants 30 stalks, each plant yields 30 oz of beans. When she plants 34 stalks, each plant produces 28 oz of beans. Find a linear relationships in the form $y = mn + b$ that gives the yield when n stalks are planted.

47. Which of the following tables could represent a linear function? For each that could be linear, find a linear equation models the data.

x	*g(x)*
0	5
5	-10
10	-25
15	-40

x	*h(x)*
0	5
5	30
10	105
15	230

x	*f(x)*
0	-5
5	20
10	45
15	70

x	*k(x)*
5	13
10	28
20	58
25	73

48. Which of the following tables could represent a linear function? For each that could be linear, find a linear equation models the data.

x	*g(x)*
0	6
2	-19
4	-44
6	-69

x	*h(x)*
2	13
4	23
8	43
10	53

x	*f(x)*
2	-4
4	16
6	36
8	56

x	*k(x)*
0	6
2	31
6	106
8	231

49. While speaking on the phone to a friend in Oslo, Norway, you learned that the current temperature there was -23 Celsius (-23°C). After the phone conversation, you wanted to convert this temperature to Fahrenheit degrees, °F, but you could not find a reference with the correct formulas. You then remembered that the relationship between °F and °C is linear. [UW]
   a. Using this and the knowledge that 32°F = 0 °C and 212 °F = 100 °C, find an equation that computes Celsius temperature in terms of Fahrenheit; i.e. an equation of the form C = "an expression involving only the variable F."
   b. Likewise, find an equation that computes Fahrenheit temperature in terms of Celsius temperature; i.e. an equation of the form F = "an expression involving only the variable C."
   c. How cold was it in Oslo in °F?