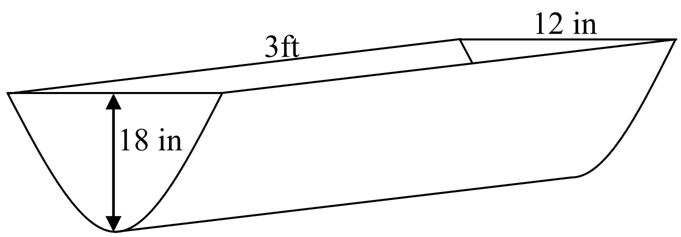
Section 3.5 Inverses and Radical Functions

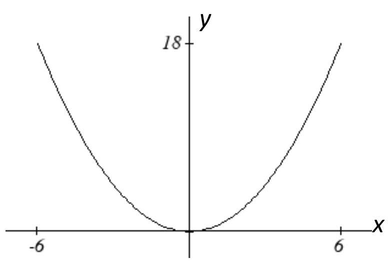
In this section, we will explore the inverses of polynomial and rational functions, and in particular the radical functions that arise in the process.

Example 1

A water runoff collector is built in the shape of a parabolic trough as shown below. Find the surface area of the water in the trough as a function of the depth of the water.



Since it will be helpful to have an equation for the parabolic cross-sectional shape, we will impose a coordinate system at the cross section, with x measured horizontally and y measured vertically, with the origin at the vertex of the parabola.



From this we find an equation for the parabolic shape. Since we placed the origin at the vertex of the parabola, we know the equation will have form $y(x) = ax^2$. Our equation will need to pass through the point (6,18), from which we can solve for the stretch factor a:

$$18 = a6^2$$

$$a = \frac{18}{36} = \frac{1}{2}$$

Our parabolic cross section has equation $y(x) = \frac{1}{2}x^2$

Since we are interested in the surface area of the water, we are interested in determining the width at the top of the water as a function of the water depth. For any depth y the width will be given by $2x$, so we need to solve the equation above for x. However notice that the original function is not one-to-one, and indeed given any output there are two inputs that produce the same output, one positive and one negative.

To find an inverse, we can restrict our original function to a limited domain on which it *is* one-to-one. In this case, it makes sense to restrict ourselves to positive x values. On this domain, we can find an inverse by solving for the input variable:

$$y = \frac{1}{2}x^2$$

$$2y = x^2$$

$$x = \pm\sqrt{2y}$$

This is not a function as written. Since we are limiting ourselves to positive x values, we eliminate the negative solution, giving us the inverse function we're looking for

$$x(y) = \sqrt{2y}$$

Since x measures from the center out, the entire width of the water at the top will be $2x$. Since the trough is 3 feet (36 inches) long, the surface area will then be $36(2x)$, or in terms of y:

$$Area = 72x = 72\sqrt{2y}$$

The previous example illustrated two important things:

1) When finding the inverse of a quadratic, we have to limit ourselves to a domain on which the function is one-to-one.
2) The inverse of a quadratic function is a square root function. Both are toolkit functions and different types of power functions.

Functions involving roots are often called **radical functions**.

Example 2

Find the inverse of $f(x) = (x-2)^2 - 3 = x^2 - 4x + 1$

From the transformation form of the function, we can see this is a transformed quadratic with vertex at (2,-3) that opens upwards. Since the graph will be decreasing on one side of the vertex, and increasing on the other side, we can restrict this function to a domain on which it will be one-to-one by limiting the domain to $x \geq 2$.

To find the inverse, we will use the vertex form of the quadratic. We start by replacing the *f(x)* with a simple variable y, then solve for x.

$y = (x-2)^2 - 3$	Add 3 to both sides
$y + 3 = (x-2)^2$	Take the square root
$\pm\sqrt{y+3} = x - 2$	Add 2 to both sides
$2 \pm \sqrt{y+3} = x$	

Of course, as written this is not a function. Since we restricted our original function to a domain of $x \geq 2$, the outputs of the inverse should be the same, telling us to utilize the + case:

$$x = f^{-1}(y) = 2 + \sqrt{y+3}$$

If the quadratic had not been given in vertex form, rewriting it into vertex form is probably the best approach. Alternatively, we could have taken the standard equation and rewritten it equal to zero:

$$0 = x^2 - 4x + 1 - y$$

We would then be able to use the quadratic formula with $a = 1$, $b = -4$, and $c = (1 - y)$, resulting in the same solutions we found above:

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1-y)}}{2} = 2 \pm \frac{\sqrt{12 + 4y}}{2} = 2 \pm \sqrt{3 + y}$$

Try it Now

1. Find the inverse of the function $f(x) = x^2 + 1$, on the domain $x \geq 0$.

While it is not possible to find an inverse of most polynomial functions, some other basic polynomials are invertible.

Example 3

Find the inverse of the function $f(x) = 5x^3 + 1$.

This is a transformation of the basic cubic toolkit function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for x

$$y = 5x^3 + 1$$

$$y - 1 = 5x^3$$

$$\frac{y-1}{5} = x^3$$

$$x = f^{-1}(y) = \sqrt[3]{\frac{y-1}{5}}$$

Notice that this inverse is also a transformation of a power function with a fractional power, $x^{1/3}$.

Try it Now

2. Which toolkit functions have inverse functions without restricting their domain?

Besides being important as an inverse function, radical functions are common in important physical models.

Example 4

The velocity, v in feet per second, of a car that slammed on its brakes can be determined based on the length of skid marks that the tires left on the ground. This relationship is given by

$$v(d) = \sqrt{2gfd}$$

In this formula, g represents acceleration due to gravity (32 ft/sec^2), d is the length of the skid marks in feet, and f is a constant representing the friction of the surface. A car lost control on wet asphalt, with a friction coefficient of 0.5, leaving 200 foot skid marks. How fast was the car travelling when it lost control?

Using the given values of $f = 0.5$ and $d = 200$, we can evaluate the given formula:
$v(200) = \sqrt{2(32)(0.5)(200)} = 80 \text{ ft/sec}$, which is about 54.5 miles per hour.

When radical functions are composed with other functions, determining domain can become more complicated.

Example 5

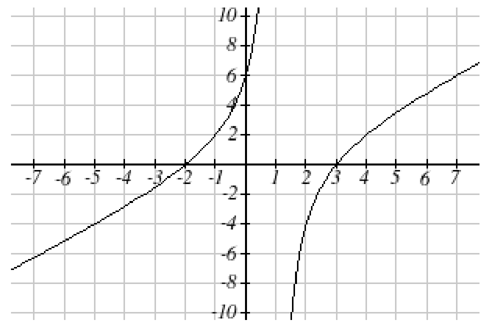
Find the domain of the function $f(x) = \sqrt{\dfrac{(x+2)(x-3)}{(x-1)}}$.

Since a square root is only defined when the quantity under the radical is non-negative, we need to determine where $\dfrac{(x+2)(x-3)}{(x-1)} \geq 0$. A rational function can change signs (change from positive to negative or vice versa) at horizontal intercepts and at vertical asymptotes. For this equation, the graph could change signs at x = -2, 1, and 3.

To determine on which intervals the rational expression is positive, we could evaluate the expression at test values, or sketch a graph. While both approaches work equally well, for this example we will use a graph.

This function has two horizontal intercepts, both of which exhibit linear behavior, where the graph will pass through the intercept. There is one vertical asymptote, corresponding to a linear factor, leading to a behavior similar to the basic reciprocal toolkit function. There is a vertical intercept at (0, 6). This graph does not have a horizontal asymptote, since the degree of the numerator is larger than the degree of the denominator.

From the vertical intercept and horizontal intercept at $x = -2$, we can sketch the left side of the graph. From the behavior at the asymptote, we can sketch the right side of the graph.



From the graph, we can now tell on which intervals this expression will be non-negative, so the original function *f(x)* will be defined. *f(x)* has domain $-2 \le x < 1 \; or \; x \ge 3$, or in interval notation, $[-2,1) \cup [3,\infty)$.

Like with finding inverses of quadratic functions, it is sometimes desirable to find the inverse of a rational function, particularly of rational functions that are the ratio of linear functions, such as our concentration examples.

Example 6

The function $C(n) = \dfrac{20 + 0.4n}{100 + n}$ was used in the previous section to represent the concentration of an acid solution after n mL of 40% solution has been added to 100 mL of a 20% solution. We might want to be able to determine instead how much 40% solution has been added based on the current concentration of the mixture.

To do this, we would want the inverse of this function:

$C = \dfrac{20 + 0.4n}{100 + n}$	multiply both sides by the denominator
$C(100 + n) = 20 + 0.4n$	distribute
$100C + Cn = 20 + 0.4n$	group everything with n on one side
$100C - 20 = 0.4n - Cn$	factor out n
$100C - 20 = (0.4 - C)n$	divide to find the inverse
$n(C) = \dfrac{100C - 20}{0.4 - C}$	

If, for example, we wanted to know how many mL of 40% solution need to be added to obtain a concentration of 35%, we can simply evaluate the inverse rather than solving an equation involving the original function:

$n(0.35) = \dfrac{100(0.35) - 20}{0.4 - 0.35} = \dfrac{15}{0.05} = 300$ mL of 40% solution would need to be added.

Try it Now

3. Find the inverse of the function $f(x) = \dfrac{x+3}{x-2}$.

Important Topics of this Section

Imposing a coordinate system
Finding an inverse function
    Restricting the domain
Invertible toolkit functions
Radical Functions
Inverses of rational functions

Try it Now Answers

1. $x = f^{-1}(y) = \sqrt{y-1}$
2. identity, cubic, square root, cube root
3. $f^{-1}(y) = \dfrac{2y+3}{y-1}$

Section 3.5 Exercises

For each function, find a domain on which the function is one-to-one and non-decreasing, then find an inverse of the function on this domain.

1. $f(x)=(x-4)^2$
2. $f(x)=(x+2)^2$
3. $f(x)=12-x^2$
4. $f(x)=9-x^2$
5. $f(x)=3x^3+1$
6. $f(x)=4-2x^3$

Find the inverse of each function.

7. $f(x)=9+\sqrt{4x-4}$
8. $f(x)=\sqrt{6x-8}+5$
9. $f(x)=9+2\sqrt[3]{x}$
10. $f(x)=3-\sqrt[3]{x}$
11. $f(x)=\dfrac{2}{x+8}$
12. $f(x)=\dfrac{3}{x-4}$
13. $f(x)=\dfrac{x+3}{x+7}$
14. $f(x)=\dfrac{x-2}{x+7}$
15. $f(x)=\dfrac{3x+4}{5-4x}$
16. $f(x)=\dfrac{5x+1}{2-5x}$

Police use the formula $v=\sqrt{20L}$ to estimate the speed of a car, v, in miles per hour, based on the length, L, in feet, of its skid marks when suddenly braking on a dry, asphalt road.

17. At the scene of an accident, a police officer measures a car's skid marks to be 215 feet long. Approximately how fast was the car traveling?

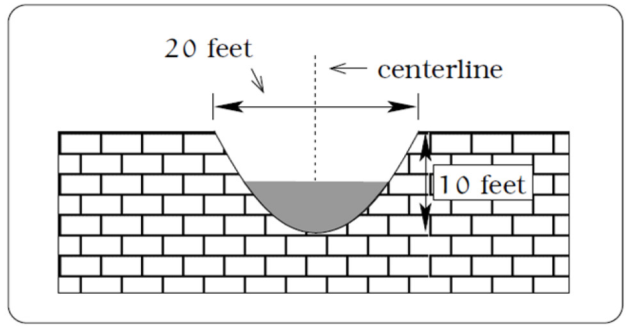
18. At the scene of an accident, a police officer measures a car's skid marks to be 135 feet long. Approximately how fast was the car traveling?

The formula $v=\sqrt{2.7r}$ models the maximum safe speed, v, in miles per hour, at which a car can travel on a curved road with radius of curvature r, in feet.

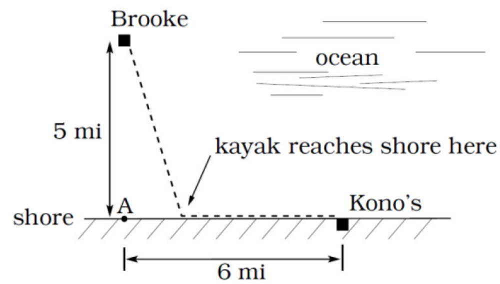
19. A highway crew measures the radius of curvature at an exit ramp on a highway as 430 feet. What is the maximum safe speed?

20. A highway crew measures the radius of curvature at a tight corner on a highway as 900 feet. What is the maximum safe speed?

21. A drainage canal has a cross-section in the shape of a parabola. Suppose that the canal is 10 feet deep and 20 feet wide at the top. If the water depth in the ditch is 5 feet, how wide is the surface of the water in the ditch? [UW]



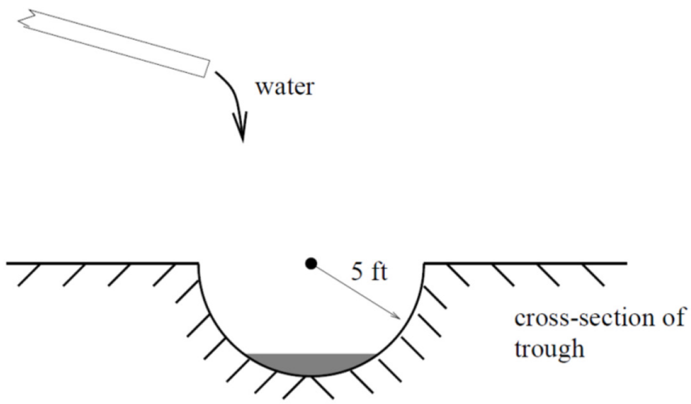
22. Brooke is located 5 miles out from the nearest point *A* along a straight shoreline in her sea kayak. Hunger strikes and she wants to make it to Kono's for lunch; see picture. Brooke can paddle 2 mph and walk 4 mph. [UW]



    a. If she paddles along a straight line course to the shore, find an expression that computes the total time to reach lunch in terms of the location where Brooke beaches her kayak.
    b. Determine the total time to reach Kono's if she paddles directly to the point *A*.
    c. Determine the total time to reach Kono's if she paddles directly to Kono's.
    d. Do you think your answer to b or c is the minimum time required for Brooke to reach lunch?
    e. Determine the total time to reach Kono's if she paddles directly to a point on the shore half way between point A and Kono's. How does this time compare to the times in parts b or c? Do you need to modify your answer to part d?

23. Clovis is standing at the edge of a dropoff, which slopes 4 feet downward from him for every 1 horizontal foot. He launches a small model rocket from where he is standing. With the origin of the coordinate system located where he is standing, and the *x*-axis extending horizontally, the path of the rocket is described by the formula $y = -2x^2 + 120x$. [UW]
    a. Give a function $h = f(x)$ relating the height *h* of the rocket above the sloping ground to its *x*-coordinate.
    b. Find the maximum height of the rocket above the sloping ground. What is its *x*-coordinate when it is at its maximum height?
    c. Clovis measures the height *h* of the rocket above the sloping ground while it is going up. Give a function $x = g(h)$ relating the *x*-coordinate of the rocket to *h*.
    d. Does the function from (c) still work when the rocket is going down? Explain.

24. A trough has a semicircular cross section with a radius of 5 feet. Water starts flowing into the trough in such a way that the depth of the water is increasing at a rate of 2 inches per hour. [UW]



a. Give a function $w = f(t)$ relating the width w of the surface of the water to the time t, in hours. Make sure to specify the domain and compute the range too.
b. After how many hours will the surface of the water have width of 6 feet?
c. Give a function $t = f^{-1}(w)$ relating the time to the width of the surface of the water. Make sure to specify the domain and compute the range too.