

Section 2.3 Modeling with Linear Functions

When modeling scenarios with a linear function and solving problems involving quantities changing linearly, we typically follow the same problem solving strategies that we would use for any type of function:

Problem solving strategy

- 1) Identify changing quantities, and then carefully and clearly define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
- 2) Carefully read the problem to identify important information. Look for information giving values for the variables, or values for parts of the functional model, like slope and initial value.
- 3) Carefully read the problem to identify what we are trying to find, identify, solve, or interpret.
- 4) Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table or even finding a formula for the function being used to model the problem.
- 5) When needed, find a formula for the function.
- 6) Solve or evaluate using the formula you found for the desired quantities.
- 7) Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
- 8) Clearly convey your result using appropriate units, and answer in full sentences when appropriate.

Example 1

Emily saved up \$3500 for her summer visit to Seattle. She anticipates spending \$400 each week on rent, food, and fun. Find and interpret the horizontal intercept and determine a reasonable domain and range for this function.

In the problem, there are two changing quantities: time and money. The amount of money she has remaining while on vacation depends on how long she stays. We can define our variables, including units.

Output: M , money remaining, in dollars

Input: t , time, in weeks

Reading the problem, we identify two important values. The first, \$3500, is the initial value for M . The other value appears to be a rate of change – the units of dollars per week match the units of our output variable divided by our input variable. She is spending money each week, so you should recognize that the amount of money remaining is decreasing each week and the slope is negative.

To answer the first question, looking for the horizontal intercept, it would be helpful to have an equation modeling this scenario. Using the intercept and slope provided in the problem, we can write the equation: $M(t) = 3500 - 400t$.

To find the horizontal intercept, we set the output to zero, and solve for the input:

$$0 = 3500 - 400t$$

$$t = \frac{3500}{400} = 8.75$$

The horizontal intercept is 8.75 weeks. Since this represents the input value where the output will be zero, interpreting this, we could say: Emily will have no money left after 8.75 weeks.

When modeling any real life scenario with functions, there is typically a limited domain over which that model will be valid – almost no trend continues indefinitely. In this case, it certainly doesn't make sense to talk about input values less than zero. It is also likely that this model is not valid after the horizontal intercept (unless Emily's going to start using a credit card and go into debt).

The domain represents the set of input values and so the reasonable domain for this function is $0 \leq t \leq 8.75$.

However, in a real world scenario, the rental might be weekly or nightly. She may not be able to stay a partial week and so all options should be considered. Emily could stay in Seattle for 0 to 8 full weeks (and a couple of days), but would have to go into debt to stay 9 full weeks, so restricted to whole weeks, a reasonable domain without going in to debt would be $0 \leq t \leq 8$, or $0 \leq t \leq 9$ if she went into debt to finish out the last week.

The range represents the set of output values and she starts with \$3500 and ends with \$0 after 8.75 weeks so the corresponding range is $0 \leq M(t) \leq 3500$.

If we limit the rental to whole weeks however, if she left after 8 weeks because she didn't have enough to stay for a full 9 weeks, she would have $M(8) = 3500 - 400(8) = \$300$ dollars left after 8 weeks, giving a range of $300 \leq M(t) \leq 3500$. If she wanted to stay the full 9 weeks she would be \$100 in debt giving a range of $-100 \leq M(t) \leq 3500$.

Most importantly remember that domain and range are tied together, and what ever you decide is most appropriate for the domain (the independent variable) will dictate the requirements for the range (the dependent variable).

Example 2

Jamal is choosing between two moving companies. The first, U-Haul, charges an up-front fee of \$20, then 59 cents a mile. The second, Budget, charges an up-front fee of \$16, then 63 cents a mile³. When will U-Haul be the better choice for Jamal?

The two important quantities in this problem are the cost, and the number of miles that are driven. Since we have two companies to consider, we will define two functions:

Input: m , miles driven

Outputs:

$Y(m)$: cost, in dollars, for renting from U-Haul

$B(m)$: cost, in dollars, for renting from Budget

Reading the problem carefully, it appears that we were given an initial cost and a rate of change for each company. Since our outputs are measured in dollars but the costs per mile given in the problem are in cents, we will need to convert these quantities to match our desired units: \$0.59 a mile for U-Haul, and \$0.63 a mile for Budget.

Looking to what we're trying to find, we want to know when U-Haul will be the better choice. Since all we have to make that decision from is the costs, we are looking for when U-Haul will cost less, or when $Y(m) < B(m)$. The solution pathway will lead us to find the equations for the two functions, find the intersection, then look to see where the $Y(m)$ function is smaller. Using the rates of change and initial charges, we can write the equations:

$$Y(m) = 20 + 0.59m$$

$$B(m) = 16 + 0.63m$$

These graphs are sketched to the right, with $Y(m)$ drawn dashed.

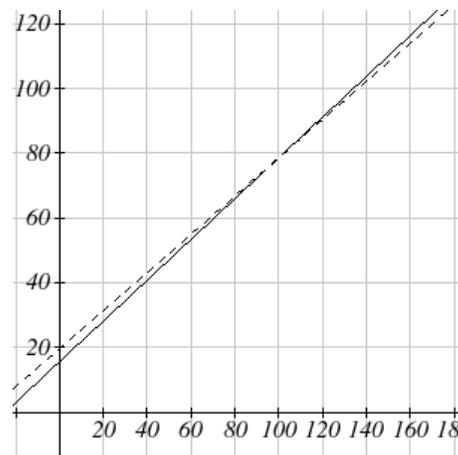
To find the intersection, we set the equations equal and solve:

$$Y(m) = B(m)$$

$$20 + 0.59m = 16 + 0.63m$$

$$4 = 0.04m$$

$$m = 100$$



This tells us that the cost from the two companies will be the same if 100 miles are driven. Either by looking at the graph, or noting that $Y(m)$ is growing at a slower rate, we can conclude that U-Haul will be the cheaper price when more than 100 miles are driven.

³ Rates retrieved Aug 2, 2010 from <http://www.budgettruck.com> and <http://www.uhaul.com/>

Example 3

A town's population has been growing linearly. In 2004 the population was 6,200. By 2009 the population had grown to 8,100. If this trend continues,

- Predict the population in 2013
- When will the population reach 15000?

The two changing quantities are the population and time. While we could use the actual year value as the input quantity, doing so tends to lead to very ugly equations, since the vertical intercept would correspond to the year 0, more than 2000 years ago!

To make things a little nicer, and to make our lives easier too, we will define our input as years since 2004:

Input: t , years since 2004

Output: $P(t)$, the town's population

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to $t = 0$, giving the point $(0, 6200)$. Notice that through our clever choice of variable definition, we have "given" ourselves the vertical intercept of the function. The year 2009 would correspond to $t = 5$, giving the point $(5, 8100)$.

To predict the population in 2013 ($t = 9$), we would need an equation for the population. Likewise, to find when the population would reach 15000, we would need to solve for the input that would provide an output of 15000. Either way, we need an equation. To find it, we start by calculating the rate of change:

$$m = \frac{8100 - 6200}{5 - 0} = \frac{1900}{5} = 380 \text{ people per year}$$

Since we already know the vertical intercept of the line, we can immediately write the equation:

$$P(t) = 6200 + 380t$$

To predict the population in 2013, we evaluate our function at $t = 9$

$$P(9) = 6200 + 380(9) = 9620$$

If the trend continues, our model predicts a population of 9,620 in 2013.

To find when the population will reach 15,000, we can set $P(t) = 15000$ and solve for t .

$$15000 = 6200 + 380t$$

$$8800 = 380t$$

$$t \approx 23.158$$

Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

Example 4

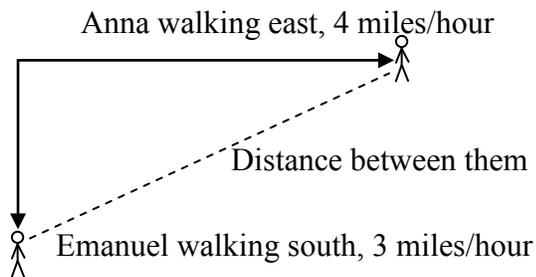
Anna and Emanuel start at the same intersection. Anna walks east at 4 miles per hour while Emanuel walks south at 3 miles per hour. They are communicating with a two-way radio with a range of 2 miles. How long after they start walking will they fall out of radio contact?

In essence, we can partially answer this question by saying they will fall out of radio contact when they are 2 miles apart, which leads us to ask a new question: how long will it take them to be 2 miles apart?

In this problem, our changing quantities are time and the two peoples' positions, but ultimately we need to know how long will it take for them to be 2 miles apart. We can see that time will be our input variable, so we'll define

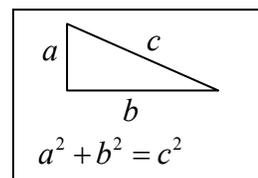
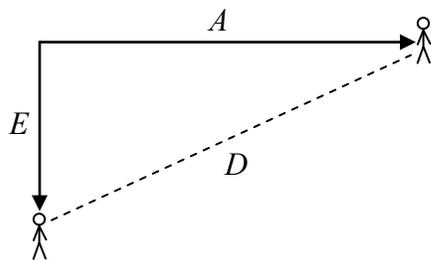
Input: t , time in hours.

Since it is not obvious how to define our output variables, we'll start by drawing a picture.



Because of the complexity of this question, it may be helpful to introduce some intermediary variables. These are quantities that we aren't directly interested in, but seem important to the problem. For this problem, Anna's and Emanuel's distances from the starting point seem important. To notate these, we are going to define a coordinate system, putting the "starting point" at the intersection where they both started, then we're going to introduce a variable, A , to represent Anna's position, and define it to be a measurement from the starting point, in the eastward direction. Likewise, we'll introduce a variable, E , to represent Emanuel's position, measured from the starting point in the southward direction. Note that in defining the coordinate system we specified both the origin, or starting point, of the measurement, as well as the direction of measure.

While we're at it, we'll define a third variable, D , to be the measurement of the distance between Anna and Emanuel. Showing the variables on the picture is often helpful: Looking at the variables on the picture, we remember we need to know how long it takes for D , the distance between them, to equal 2 miles.



Seeing this picture we remember that in order to find the distance between the two, we can use the Pythagorean Theorem, a property of right triangles.

From here, we can now look back at the problem for relevant information. Anna is walking 4 miles per hour, and Emanuel is walking 3 miles per hour, which are rates of change. Using those, we can write formulas for the distance each has walked.

They both start at the same intersection and so when $t = 0$, the distance travelled by each person should also be 0, so given the rate for each, and the initial value for each, we get:

$$A(t) = 4t$$

$$E(t) = 3t$$

Using the Pythagorean theorem we get:

$$D(t)^2 = A(t)^2 + E(t)^2$$

$$D(t)^2 = (4t)^2 + (3t)^2 = 16t^2 + 9t^2 = 25t^2$$

$$D(t) = \pm\sqrt{25t^2} = \pm 5|t|$$

substitute in the function formulas

solve for $D(t)$ using the square root

Since in this scenario we are only considering positive values of t and our distance $D(t)$ will always be positive, we can simplify this answer to $D(t) = 5t$

Interestingly, the distance between them is also a linear function. Using it, we can now answer the question of when the distance between them will reach 2 miles:

$$D(t) = 2$$

$$5t = 2$$

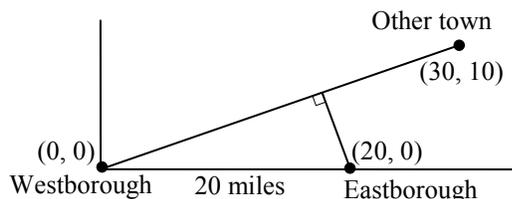
$$t = \frac{2}{5} = 0.4$$

They will fall out of radio contact in 0.4 hours, or 24 minutes.

Example 5

There is currently a straight road leading from the town of Westborough to a town 30 miles east and 10 miles north. Partway down this road, it junctions with a second road, perpendicular to the first, leading to the town of Eastborough. If the town of Eastborough is located 20 miles directly east of the town of Westborough, how far is the road junction from Westborough?

It might help here to draw a picture of the situation. It would then be helpful to introduce a coordinate system. While we could place the origin anywhere, placing it at Westborough seems convenient. This puts the other town at coordinates $(30, 10)$, and Eastborough at $(20, 0)$.



Using this point along with the origin, we can find the slope of the line from

Westborough to the other town: $m = \frac{10-0}{30-0} = \frac{1}{3}$. This gives the equation of the road

from Westborough to the other town to be $W(x) = \frac{1}{3}x$.

From this, we can determine the perpendicular road to Eastborough will have slope $m = -3$. Since the town of Eastborough is at the point $(20, 0)$, we can find the equation:

$$E(x) = -3x + b$$

plug in the point $(20, 0)$

$$0 = -3(20) + b$$

$$b = 60$$

$$E(x) = -3x + 60$$

We can now find the coordinates of the junction of the roads by finding the intersection of these lines. Setting them equal,

$$\frac{1}{3}x = -3x + 60$$

$$\frac{10}{3}x = 60$$

$$10x = 180$$

$$x = 18$$

Substituting this back into $W(x)$

$$y = W(18) = \frac{1}{3}(18) = 6$$

The roads intersect at the point $(18, 6)$. Using the distance formula, we can now find the distance from Westborough to the junction:

$$\text{dist} = \sqrt{(18-0)^2 + (6-0)^2} \approx 18.934 \text{ miles.}$$

Important Topics of this Section**The problem solving process**

- 1) Identify changing quantities, and then carefully and clearly define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
- 2) Carefully read the problem to identify important information. Look for information giving values for the variables, or values for parts of the functional model, like slope and initial value.
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Section 2.3 Exercises

1. In 2004, a school population was 1001. By 2008 the population had grown to 1697. Assume the population is changing linearly.
 - a. How much did the population grow between the year 2004 and 2008?
 - b. How long did it take the population to grow from 1001 students to 1697 students?
 - c. What is the average population growth per year?
 - d. What was the population in the year 2000?
 - e. Find an equation for the population, P , of the school t years after 2000.
 - f. Using your equation, predict the population of the school in 2011.

2. In 2003, a town's population was 1431. By 2007 the population had grown to 2134. Assume the population is changing linearly.
 - a. How much did the population grow between the year 2003 and 2007?
 - b. How long did it take the population to grow from 1431 people to 2134?
 - c. What is the average population growth per year?
 - d. What was the population in the year 2000?
 - e. Find an equation for the population, P , of the town t years after 2000.
 - f. Using your equation, predict the population of the town in 2014.

3. A phone company has a monthly cellular plan where a customer pays a flat monthly fee and then a certain amount of money per minute used on the phone. If a customer uses 410 minutes, the monthly cost will be \$71.50. If the customer uses 720 minutes, the monthly cost will be \$118.
 - a. Find a linear equation for the monthly cost of the cell plan as a function of x , the number of monthly minutes used.
 - b. Interpret the slope and vertical intercept of the equation.
 - c. Use your equation to find the total monthly cost if 687 minutes are used.

4. A phone company has a monthly cellular data plan where a customer pays a flat monthly fee and then a certain amount of money per megabyte (MB) of data used on the phone. If a customer uses 20 MB, the monthly cost will be \$11.20. If the customer uses 130 MB, the monthly cost will be \$17.80.
 - a. Find a linear equation for the monthly cost of the data plan as a function of x , the number of MB used.
 - b. Interpret the slope and vertical intercept of the equation.
 - c. Use your equation to find the total monthly cost if 250 MB are used.

5. In 1991, the moose population in a park was measured to be 4360. By 1999, the population was measured again to be 5880. If the population continues to change linearly,
 - a. Find a formula for the moose population, P .
 - b. What does your model predict the moose population to be in 2003?

6. In 2003, the owl population in a park was measured to be 340. By 2007, the population was measured again to be 285. If the population continues to change linearly,
 - a. Find a formula for the owl population, P .
 - b. What does your model predict the owl population to be in 2012?

7. The Federal Helium Reserve held about 16 billion cubic feet of helium in 2010, and is being depleted by about 2.1 billion cubic feet each year.
 - a. Give a linear equation for the remaining federal helium reserves, R , in terms of t , the number of years since 2010.
 - b. In 2015, what will the helium reserves be?
 - c. If the rate of depletion doesn't change, when will the Federal Helium Reserve be depleted?

8. Suppose the world's current oil reserves are 1820 billion barrels. If, on average, the total reserves is decreasing by 25 billion barrels of oil each year:
 - a. Give a linear equation for the remaining oil reserves, R , in terms of t , the number of years since now.
 - b. Seven years from now, what will the oil reserves be?
 - c. If the rate of depletion isn't change, when will the world's oil reserves be depleted?

9. You are choosing between two different prepaid cell phone plans. The first plan charges a rate of 26 cents per minute. The second plan charges a monthly fee of \$19.95 *plus* 11 cents per minute. How many minutes would you have to use in a month in order for the second plan to be preferable?

10. You are choosing between two different window washing companies. The first charges \$5 per window. The second charges a base fee of \$40 plus \$3 per window. How many windows would you need to have for the second company to be preferable?

11. When hired at a new job selling jewelry, you are given two pay options:
Option A: Base salary of \$17,000 a year, with a commission of 12% of your sales
Option B: Base salary of \$20,000 a year, with a commission of 5% of your sales
How much jewelry would you need to sell for option A to produce a larger income?

12. When hired at a new job selling electronics, you are given two pay options:
 Option A: Base salary of \$14,000 a year, with a commission of 10% of your sales
 Option B: Base salary of \$19,000 a year, with a commission of 4% of your sales
 How much electronics would you need to sell for option A to produce a larger income?
13. Find the area of a triangle bounded by the y axis, the line $f(x) = 9 - \frac{6}{7}x$, and the line perpendicular to $f(x)$ that passes through the origin.
14. Find the area of a triangle bounded by the x axis, the line $f(x) = 12 - \frac{1}{3}x$, and the line perpendicular to $f(x)$ that passes through the origin.
15. Find the area of a parallelogram bounded by the y axis, the line $x = 3$, the line $f(x) = 1 + 2x$, and the line parallel to $f(x)$ passing through $(2, 7)$
16. Find the area of a parallelogram bounded by the x axis, the line $g(x) = 2$, the line $f(x) = 3x$, and the line parallel to $f(x)$ passing through $(6, 1)$
17. If $b > 0$ and $m < 0$, then the line $f(x) = b + mx$ cuts off a triangle from the first quadrant. Express the area of that triangle in terms of m and b . [UW]
18. Find the value of m so the lines $f(x) = mx + 5$ and $g(x) = x$ and the y -axis form a triangle with an area of 10. [UW]
19. The median home values in Mississippi and Hawaii (adjusted for inflation) are shown below. If we assume that the house values are changing linearly,

Year	Mississippi	Hawaii
1950	25200	74400
2000	71400	272700

- In which state have home values increased at a higher rate?
- If these trends were to continue, what would be the median home value in Mississippi in 2010?
- If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd)

20. The median home value ins Indiana and Alabama (adjusted for inflation) are shown below. If we assume that the house values are changing linearly,

Year	Indiana	Alabama
1950	37700	27100
2000	94300	85100

- In which state have home values increased at a higher rate?
 - If these trends were to continue, what would be the median home value in Indiana in 2010?
 - If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd)
21. Pam is taking a train from the town of Rome to the town of Florence. Rome is located 30 miles due West of the town of Paris. Florence is 25 miles East, and 45 miles North of Rome. On her trip, how close does Pam get to Paris? [UW]
22. You're flying from Joint Base Lewis-McChord (JBLM) to an undisclosed location 226 km south and 230 km east. Mt. Rainier is located approximately 56 km east and 40 km south of JBLM. If you are flying at a constant speed of 800 km/hr, how long after you depart JBLM will you be the closest to Mt. Rainier?